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On the perturbation of a uniform tiling with resistors

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The perturbation of a uniformly tiled resistor network by adding an edge (a resistor) to the network is considered. The two-point resistance on the perturbed tiling in terms of that on the perfect tiling is obtained using Green's function. Some theoretical results are presented for an infinite *modified* square lattice. These results are confirmed experimentally by constructing an actual resistor lattice of size 13×13.

Keywords: Perturbation; uniform tiling; two-point resistance; Green's function; modified square lattice.

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1. Introduction

One of the most important and interesting problems in electric circuit theory is the computation of the equivalent resistance between any pair of nodes in a resistor electrical network, which is a classic problem in electric circuit theory studied

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by numerous authors for more than 170 years.[1] An old analysis method is the Kirchhoff's laws[1] that can be applied in principle to any resistor network, however, with increasing the size of the network solving the problem becomes difficult to analyze. Several techniques have been developed to solve this problem for infinite networks.[2–7] The most elegant and efficient method to study this problem is based on the lattice Green's function, which has been introduced for perfect resistor networks[6,7] and for perturbed networks.[8] Following these pioneering works, numerous studies of the resistance and capacitance problems had been published.[9–22]

Wu[23] established a theorem to evaluate the two-node resistance on a finite regular resistor network using the Laplacian approach. The Laplacian method has resolved a variety of resistor networks with all kinds of geometry after a slight modification of the formulation.[24–28] In recent years, a new Recursion–Transform (RT) method is created by Tan,[29] which is an alternative direct approach to calculate the resistances of the resistor networks, and express the resistance directly in a single summation. With the further development of the RT method, many resistor networks of various topologies are resolved, such as a cobweb model,[30–35] a globe network,[35,36] a fan network,[34] and a resistor network with arbitrary boundaries.[37,38]

In previous work,[11] using the lattice Green's function technique,[8] the effective resistance of a perturbed lattice is obtained by the insertion of an extra resistance connected between any pairs of nodes in the perfect lattice in which each unit cell has only one lattice point. In this paper, we extend the formulation of Ref. 11 to the perturbed tiling of d-dimensional space with resistors in which each unit cell has any number of lattice sites. Figure 1 shows a perturbed modified square lattice as an example for the perturbed tiling in two dimensions.

2. Two-Point Resistance on the Perturbed Tiling

In this section, we determine the two-point resistance on the perturbed tiling that is obtained by connecting an additional resistor between any two nodes in the perfect uniform tiling. First, we recall some definitions and formulae that we use in this paper.

Consider an infinite perfect lattice structure of resistors that is a uniform tiling of d-dimensional space with identical resistances R. The lattice points are given by $\mathbf{r}_i = i_1\mathbf{a}_1 + i_2\mathbf{a}_2 + \cdots + i_d\mathbf{a}_d$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$ are the unit cell vectors in the d-dimensional space and $i_1, i_2, \ldots, i_d$ are arbitrary integers. Assume that in each unit cell there are s lattice sites labeled by $\alpha = 1, 2, \ldots, s$. If $\mathbf{r}_{i\alpha} = \{\mathbf{r}_i; \alpha\}$ denotes any lattice point, then $I_{i\alpha} = I(\mathbf{r}_{i\alpha}) = I_\alpha(\mathbf{r}_i)$ and $V_{i\alpha} = V(\mathbf{r}_{i\alpha}) = V_\alpha(\mathbf{r}_i)$ denote the current and potential at point $\mathbf{r}_{i\alpha}$, respectively. Also, the current and potential can be represented in Dirac notation as

$$I_\alpha(\mathbf{r}_i) = \langle i\alpha|I\rangle, \quad V_\alpha(\mathbf{r}_i) = \langle i\alpha|V\rangle, \tag{1}$$

where $|I\rangle$, $|V\rangle$ and $|i\alpha\rangle$ are associated vectors with the current, the potential and the lattice point $\mathbf{r}_{i\alpha}$, respectively. It is supposed that $|i\alpha\rangle$ forms a complete orthonormal set:

$$\langle i\alpha | j\beta\rangle = \delta_{ij}\delta_{\alpha\beta}, \quad \sum_{i\alpha} |i\alpha\rangle\langle i\alpha| = 1. \tag{2}$$

In the same way as in Ref. 8, the resistance between sites $\mathbf{r}_{i\alpha}$ and $\mathbf{r}_{j\beta}$ in the perfect tiled lattice can be obtained as

$$R^0_{\alpha\beta}(i,j) = R\left(G^0_{\alpha\alpha}(i,i) + G^0_{\beta\beta}(j,j) - G^0_{\alpha\beta}(i,j) - G^0_{\beta\alpha}(j,i)\right), \tag{3}$$

where $G^0_{\alpha\beta}(i,j) = G^0_{\alpha\beta}(\mathbf{r}_i,\mathbf{r}_j) \equiv \langle i\alpha|G^0|j\beta\rangle$ is the lattice Green's function for the perfect tiled network.

Now if an additional resistor of resistance R_{add} is introduced between two arbitrary sites $\{\mathbf{r}_{i_0};\alpha_0\}$ and$\{\mathbf{r}_{j_0};\beta_0\}$ in a perfect periodic lattice of equal resistances R, then the current due to this resistor at site $\{\mathbf{r}_i;\mu\}$ is given by

$$\delta I_\mu(i) = \delta_{ii_0}\delta_{\mu\alpha_0}\left(\frac{V_{\alpha_0}(i_0) - V_{\beta_0}(j_0)}{R_{\text{add}}}\right) + \delta_{ij_0}\delta_{\mu\beta_0}\left(\frac{V_{\beta_0}(j_0) - V_{\alpha_0}(i_0)}{R_{\text{add}}}\right). \tag{4}$$

Using Eqs. (1) and (2), the above equation can be written as

$$\delta I_\mu(i)R = \langle i\mu|L_{\text{add}}|V\rangle, \tag{5}$$

where the L_{add} is the perturbation operator and given by

$$L_{\text{add}} = \frac{R}{R_{\text{add}}}(|i_0\alpha_0\rangle - |j_0\beta_0\rangle)(\langle i_0\alpha_0| - \langle j_0\beta_0|). \tag{6}$$

According to Ohm's and Kirchhoff's laws, the current at site $\{\mathbf{r}_i;\mu\}$ in the perturbed tiled lattice is given by

$$I_\mu(i) = \frac{-L_0V_\mu(i)}{R} + \delta I_\mu(i). \tag{7}$$

Equation (7) can be written as

$$L|V\rangle = -R|I\rangle, \tag{8}$$

where L is the Laplacian matrix for the perturbed tiled lattice:

$$L = L_0 - L_{\text{add}}. \tag{9}$$

As usual, the Green's function for the perturbed tiled lattice ($G = -L^{-1}$) is related to the Green's function for the perfect uniform tiling ($G^0 = -(L_0)^{-1}$) through the Dyson's equation[39]:

$$G = G^0 + G^0L_{\text{add}}G. \tag{10}$$

The Dyson's equation can be solved for G by iteration, one obtains an infinite geometric series

$$G = G^0 - G^0L_{\text{add}}G^0 + G^0L_{\text{add}}G^0L_{\text{add}}G^0 - G^0L_{\text{add}}G^0L_{\text{add}}G^0L_{\text{add}}G^0 + \cdots. \tag{11}$$

The sum of the geometric series in the above equation, after inserting L_{add} given in Eq. (6), is

$$G = G^0 - \frac{RG^0(|i_0\alpha_0\rangle - |j_0\beta_0\rangle)(\langle i_0\alpha_0| - \langle j_0\beta_0|)G^0}{R_{\text{add}} + R(\langle i_0\alpha_0| - \langle j_0\beta_0|)G^0(|i_0\alpha_0\rangle - |j_0\beta_0\rangle)}. \tag{12}$$

Multiplying the left-hand side by $\langle i\alpha|$ and the right-hand side by $|j\beta\rangle$ of Eq. (12) yields

$$\begin{aligned} G_{\alpha\beta}(i,j) &= G^0_{\alpha\beta}(i,j) \\ &\quad - \frac{R(G^0_{\alpha\alpha_0}(i,i_0) - G^0_{\alpha\beta_0}(i,j_0))(G^0_{\alpha_0\beta}(i_0,j) - G^0_{\beta_0\beta}(j_0,j))}{R_{\text{add}} + R(G^0_{\alpha_0\alpha_0}(i_0,i_0) + G^0_{\beta_0\beta_0}(j_0,j_0) - G^0_{\alpha_0\beta_0}(i_0,j_0) - G^0_{\beta_0\alpha_0}(j_0,i_0))}. \end{aligned} \tag{13}$$

Following the same procedures of the perfect tiled lattice,[7] the resistance between lattice points $\mathbf{r}_{i\alpha}$ and $\mathbf{r}_{j\beta}$ in the perturbed tilled lattice can be obtained as

$$R_{\alpha\beta}(i,j) = \frac{V_\alpha(i) - V_\beta(j)}{I} = R(G_{\alpha\alpha}(i,i) + G_{\beta\beta}(j,j) - G_{\alpha\beta}(i,j) - G_{\beta\alpha}(j,i)). \tag{14}$$

Substituting Eq. (13) into (14) and using (3) yields

$$R_{\alpha\beta}(i,j) = R^0_{\alpha\beta}(i,j) - \frac{(R^0_{\alpha\alpha_0}(i,i_0) + R^0_{\beta\beta_0}(j,j_0) - R^0_{\alpha\beta_0}(i,j_0) - R^0_{\beta\alpha_0}(j,i_0))^2}{4(R_{\text{add}} + R^0_{\alpha_0\beta_0}(i_0,j_0))}. \tag{15}$$

This is the main result for the two-point resistance on and infinite perturbed tilings (i.e., adding one resistor to the perfect uniform tiling) in which each unit cell has any number of lattice sites. It is worth mentioning that our result Eq. (15) differs from Eq. (27) in Ref. 8 by the fact that in this paper we added a resistor to the perfect lattice with more than one type site while in Ref. 8 a resistor was removed from the perfect lattice with only one type site in each unit cell.

3. Results and Discussion

3.1. *Theoretical results*

In this section, we present some results for *modified* square lattices. The modified square lattice is a uniform tiling of the plane as shown in Fig. 1. The unit cell has four lattice points labeled by $\alpha = A, B, C, D$ and its vectors are $\mathbf{a}_1$ and $\mathbf{a}_2$. The two-point resistance on the infinite perfect modified square network of equal resistances R can be computed using the general lattice Green's function approach given in Ref. 7 (see Appendix A).

On the perturbed modified square network, the resistance can be determined from Eq. (15). As an example, consider $R_{\text{add}} = R$ is placed between the sites

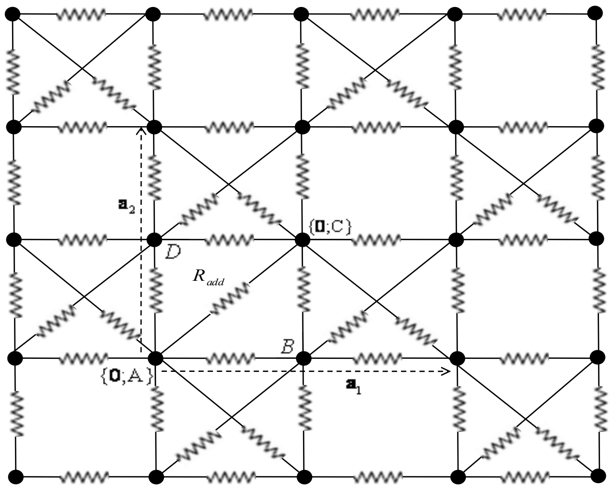

Fig. 1. A modified square lattice resistor network with an additional resistor.

Table 1. Theoretical and experimental values of the resistances $R_{\alpha\beta}(\mathbf{r}_i = \mathbf{0}, \mathbf{r}_j)$ (in unit of R) between $\{\mathbf{r}_i = \mathbf{0}; \alpha\}$ and $\{\mathbf{r}_j; \beta\}$ in the perfect and perturbed modified square lattices of resistors of value R. The additional resistor $R_{\text{add}} = R$ is placed between the sites $\{\mathbf{0}; A\}$ and $\{\mathbf{0}; C\}$ in the perfect lattice. The theoretical values are for the infinite perfect and perturbed lattices, and the experimental values are for the 13×13 perturbed lattice. The values in parentheses are the deviations from the infinite lattice values.

	Theoretical results		Experimental results
$R_{\alpha\beta}(\mathbf{r}_i = \mathbf{0}, \mathbf{r}_j)$ (in terms of R)	Infinite perfect lattice	Infinite perturbed lattice	13×13 perturbed network
R_{AB} (0,0)	0.329577	0.299877	0.302269 (0.8%)
R_{AC} (0,0)	0.409155	0.290355	0.293573 (1.11%)
R_{AD} (0,0)	0.329577	0.299877	0.304841 (1.66%)
R_{BC} (0,0)	0.329577	0.299877	0.304841 (1.66%)
R_{BD} (0,0)	0.409155	0.409155	0.415333 (1.79%)
R_{CD} (0,0)	0.329577	0.299877	0.302269 (0.8%)
R_{AA} (2,0)	0.5494132	0.525079	0.548583 (4.48%)
R_{AA} (3,0)	0.613075	0.587067	0.777328 (32.41%)

$\{\mathbf{r}_{i0} = \mathbf{0};\ \alpha_0 = A\}$ and $\{\mathbf{r}_{j0} = \mathbf{0}; \beta_0 = C\}$ in the perfect network. We calculated the resistance $R_{\alpha\beta}(\mathbf{r}_i = \mathbf{0},\ \mathbf{r}_j)$ in units of R between the origin $\{\mathbf{r}_i = \mathbf{0}; \alpha\}$ and node $\{\mathbf{r}_j; \beta\}$, and displaced them in Table 1. One can see from the table that the resistance between the lattice points $\{\mathbf{0}; A\}$ and $\{\mathbf{r}; \beta\}$ in the perturbed lattice is smaller than that between them in the corresponding perfect lattice. This is very obvious from the negativity of second term in Eq. (15).

3.2. *Experimental results*

To experimentally study our results, we constructed a finite perturbed modified square network of size (13×13) using $1 - \text{k}\Omega(\pm 10\%)$ resistors. We measured the mean resistance of the individual resistors of the network and find $R = 0.988$ kΩ. Hence, the value of the individual resistance is $R = 0.988$ kΩ.

We performed resistance measurements $R_{\alpha\beta}(\mathbf{r}_i = \mathbf{0}, \mathbf{r}_j)$ between the origin $\{\mathbf{r}_i = \mathbf{0}; \alpha\}$ and site $\{\mathbf{r}_j; \beta\}$ in the perturbed network. Table 1 displays our experimental measurements and compares them to the theoretical results for the infinite network, normalized by the individual resistance $R = 0.988$ kΩ.

It can be seen in the table that the theoretical and the experimental results are in good agreement near the network origin, but become worse as one of the sites gets closer to the boundaries of the finite network. This discrepancy is due to the finite size of the experimental network, which causes the effective resistances to be larger than the values for an infinite network.

4. Conclusion

In this work, we extended the Green's function approach[8,11] to study the two-point resistance on a perturbed network that is obtained by adding a resistor to any perfect lattice structure which is a uniform tiling of d-dimensional space. We presented some theoretical values of the resistance of the infinite perfect and perturbed modified square networks. The theoretical results for perturbed network are verified experimentally by constructing a real finite network of resistors. We found the theoretical and experimental results to be consistent within the estimated error bounds.

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Appendix A. Two-Point Resistance on the Perfect Modified Square Lattice

In this Appendix, we compute the two-point resistance on the perfect modified square lattice using general Green's function method introduced by Cserti *et al.*[7] Applying Kirchhoff's junction rule to lattice points $\{\mathbf{r}_i; \alpha = A, B, C, D\}$ and using Ohm's law, the currents at these lattice points can be written as

$$\begin{aligned} I_A(\mathbf{r}) = {} & \frac{V_A(\mathbf{r}) - V_B(\mathbf{r})}{R} + \frac{V_A(\mathbf{r}) - V_B(\mathbf{r} - \mathbf{a}_1)}{R} + \frac{V_A(\mathbf{r}) - V_C(\mathbf{r} - \mathbf{a}_1)}{R} \\ & + \frac{V_A(\mathbf{r}) - V_C(\mathbf{r} - \mathbf{a}_2)}{R} + \frac{V_A(\mathbf{r}) - V_D(\mathbf{r})}{R} + \frac{V_A(\mathbf{r}) - V_D(\mathbf{r} - \mathbf{a}_2)}{R}, \end{aligned} \tag{A.1}$$

$$\begin{aligned} I_B(\mathbf{r}) = {} & \frac{V_B(\mathbf{r}) - V_A(\mathbf{r})}{R} + \frac{V_B(\mathbf{r}) - V_A(\mathbf{r} + \mathbf{a}_1)}{R} + \frac{V_B(\mathbf{r}) - V_C(\mathbf{r})}{R} \\ & + \frac{V_B(\mathbf{r}) - V_C(\mathbf{r} - \mathbf{a}_2)}{R} + \frac{V_B(\mathbf{r}) - V_D(\mathbf{r} + \mathbf{a}_1)}{R} + \frac{V_B(\mathbf{r}) - V_D(\mathbf{r} - \mathbf{a}_2)}{R}, \end{aligned} \tag{A.2}$$

$$I_C(\mathbf{r}) = \frac{V_C(\mathbf{r}) - V_A(\mathbf{r}+\mathbf{a}_1)}{R} + \frac{V_C(\mathbf{r}) - V_A(\mathbf{r}+\mathbf{a}_2)}{R} + \frac{V_C(\mathbf{r}) - V_B(\mathbf{r})}{R} + \frac{V_C(\mathbf{r}) - V_B(\mathbf{r}+\mathbf{a}_2)}{R} + \frac{V_C(\mathbf{r}) - V_D(\mathbf{r})}{R} + \frac{V_C(\mathbf{r}) - V_D(\mathbf{r}+\mathbf{a}_1)}{R}, \quad \text{(A.3)}$$

$$I_D(\mathbf{r}) = \frac{V_D(\mathbf{r}) - V_A(\mathbf{r})}{R} + \frac{V_D(\mathbf{r}) - V_A(\mathbf{r}+\mathbf{a}_2)}{R} + \frac{V_D(\mathbf{r}) - V_B(\mathbf{r}-\mathbf{a}_1)}{R} + \frac{V_D(\mathbf{r}) - V_B(\mathbf{r}+\mathbf{a}_2)}{R} + \frac{V_D(\mathbf{r}) - V_C(\mathbf{r})}{R} + \frac{V_D(\mathbf{r}) - V_C(\mathbf{r}-\mathbf{a}_1)}{R}. \quad \text{(A.4)}$$

Assuming periodic boundary conditions, the discrete Fourier transforms of the potentials and currents are defined as

$$V_\alpha(\mathbf{k}) = \sum_{\mathbf{r}} V_\alpha(\mathbf{r}) e^{-i\mathbf{k}.\mathbf{r}}, \quad I_\alpha(\mathbf{k}) = \sum_{\mathbf{r}} I_\alpha(\mathbf{r}) e^{-i\mathbf{k}.\mathbf{r}}, \quad \text{(A.5)}$$

where $\mathbf{k}$ is the wave vector in the Fourier space and is limited to the first Brillouin zone. The general expressions for the inverse Fourier transform are given by

$$V_\alpha(\mathbf{r}) = \frac{A_0}{(2\pi)^2} \int_{-\pi/a_1}^{\pi/a_1} \int_{-\pi/a_2}^{\pi/a_2} V_\alpha(\mathbf{k}) e^{i\mathbf{k}.\mathbf{r}} dk_1 dk_2, \quad \text{(A.6)}$$

$$I_\alpha(\mathbf{r}) = \frac{A_0}{(2\pi)^2} \int_{-\pi/a_1}^{\pi/a_1} \int_{-\pi/a_2}^{\pi/a_2} I_\alpha(\mathbf{k}) e^{i\mathbf{k}.\mathbf{r}} dk_1 dk_2. \quad \text{(A.7)}$$

Substituting Eqs. (A.6) and (A.7) into (A.1)–(A.4) and making the transformation $\theta_i = \mathbf{k} \cdot \mathbf{a}_i (i = 1, 2)$ yields

$$\mathbf{L_0 V} = -R\mathbf{I}, \quad \text{(A.8)}$$

where

$$\mathbf{L}_0(\theta_1, \theta_2) = \begin{pmatrix} -6 & 1 + e^{-i\theta_1} & e^{-i\theta_1} + e^{-i\theta_2} & 1 + e^{-i\theta_2} \\ 1 + e^{i\theta_1} & -6 & 1 + e^{-i\theta_2} & e^{i\theta_1} + e^{-i\theta_2} \\ e^{i\theta_1} + e^{i\theta_2} & 1 + e^{i\theta_2} & -6 & 1 + e^{i\theta_1} \\ 1 + e^{i\theta_2} & e^{-i\theta_1} + e^{i\theta_2} & 1 + e^{-i\theta_1} & -6 \end{pmatrix} \quad \text{(A.9)}$$

is the Fourier transform of the Laplacian matrix for the perfect lattice. The lattice Green's function can be calculated from the definition:

$$\mathbf{G}_0 = -\mathbf{L}_0^{-1}. \quad \text{(A.10)}$$

Now, the resistance between any two sites on perfect modified square lattice can be calculated from the following expression

$$R^0_{\alpha\beta}(\ell_1, \ell_2) = R \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \begin{Bmatrix} G^0_{\alpha\alpha}(\theta_1, \theta_2) - G^0_{\alpha\beta} e^{-i(\ell_1\theta_1 + \ell_2\theta_2)} \\ +G^0_{\beta\beta}(\theta_1, \theta_2) - G^0_{\beta\alpha} e^{i(\ell_1\theta_1 + \ell_2\theta_2)} \end{Bmatrix}. \quad \text{(A.11)}$$

Some values of the resistances the origin $\{\mathbf{r}_i = \mathbf{0}; \alpha\}$ and site $\{\mathbf{r}_j; \beta\}$ are displayed in Table 1.

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